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The Largest Non-vanishing Degree of Graded Local Cohomology Modules

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This paper gives characterizations for the largest non-vanishing degree of the local cohomology modules of a graded ring S in terms of a reduction of S_+ and of the associated graded ring of an ideal I in terms of any reduction of I. As a consequence, this invariant can be computed explicitly for the associated graded ring of any ideal generated by a d-sequence and of ideals having small analytic deviation. © 1999 Academic Press

1. INTRODUCTION

Let $S = \oplus_{n \geq 0} S_n$ be a finitely generated standard graded ring over a local ring S_0 (not necessarily of finite length), where "standard" means S is generated over S_0 by the homogeneous elements of degree 1. For convenience we will assume that the residue field of S_0 is infinite.

Let S_+ denote the ideal generated by the elements of positive degree and M the maximal graded ideal of S. For $Q = S_+$ or M we denote by $H^i_Q(S)$ the ith local cohomology module of S with respect to Q and by $a^i_Q(S)$ the largest non-vanishing degree of $H^i_Q(S)$, where $a^i_Q(S) = -\infty$ if $H^i_Q(S) = 0$. The invariants $a^i_{S_+}(S)$ and $a^i_M(S)$ carry important information on the structure of S. For instance, the Castelnuovo–Mumford regularity

$$\operatorname{reg}(S) := \max\left\{a^i_{S_+}(S) + i \mid i \geq 0\right\}$$

gives upper bounds for the degree of the syzygies of S [EG, O1], and $a^d_M(S)$, $d = \dim S$, is equal to the a-invariant $a(S)$ of S which gives the least non-vanishing degree of the canonical module of S [GW].

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If S is a Cohen–Macaulay ring, several estimates for $a(S)$ have been established, notably, when S is the associated graded ring of an ideal (see, e.g., [SUV, AHT, U, HHK]). If S is not a Cohen–Macaulay ring, almost nothing on $a(S)$ is known. In this case, one should better estimate the larger invariant

$$a^*(S) := \max\{a_M^i(S) \mid i \geq 0\}.$$

This invariant is of independent interest. For instance, if S_0 is an artinian local ring, then $a^*(S) + 1$ gives an upper bound for the *postulation number* $p(S)$ of S which is the least integer $p \geq 0$ such that $H_S(n) = P_S(n)$ for all $n \geq p$, where $H_S(n)$ and $P_S(n)$ denote the Hilbert function and the Hilbert polynomial of S. By a result of Serre (see, e.g., [O2]) we know that

$$H_S(n) - P_S(n) = \sum_{i=0}^{d} (-1)^i l\big(H_M^i(S)_n\big).$$

In [T2] the author has shown that the Castelnuovo–Mumford regularity $\operatorname{reg}(S)$ can be characterized in terms of any reduction Q of S_+ generated by forms of degree 1. Recall that a graded ideal $Q \subseteq S_+$ is a *reduction* of S_+ if $Q_n = S_n$ for some integer n. The least integer n for which $Q_{n+1} = S_{n+1}$ is the *reduction number* of Q, denoted by $r_Q(S_+)$. A sequence $\mathbf{z} = z_1, \dots, z_v$ of homogeneous elements of S is called *filter-regular* if $x_i \notin P$ for every associated prime ideal $P \supseteq S_+$ of $(z_1, \dots, z_{i-1})$, $i = 1, \dots, v$. This is equivalent to the condition $[(z_1, \dots, z_{i-1}) : z_i]_n = (z_1, \dots, z_{i-1})_n$ for n large enough, $i = 1, \dots, v$. Let $a(\mathbf{z})$ denote the least number s for which this relation holds for $n > s$, $i = 1, \dots, v$. We can always choose a filter-regular sequence $\mathbf{z}$ of homogeneous forms of degree 1 which generates Q. Then we can show that

$$\operatorname{reg}(S) = \max\{a(\mathbf{z}), r_Q(S_+)\}.$$

This formula gives an effective tool for estimating $\operatorname{reg}(S)$. In particular, we can characterize the Castelnuovo–Mumford regularity of the associated graded ring $G = \oplus_{n \geq 0} I^n/I^{n+1}$ and the Rees algebra $R = \oplus_{n \geq 0} I^n$ of an ideal I in a local ring A in terms of any reduction of I.

In this paper we will show that there are similar characterizations for $a^*(S)$ in terms of a reduction of S_+ and for $a^*(G)$ in terms of any reduction of I.

Let $s(\mathbf{z})$ denote the least number s such that $[(z_1, \dots, z_{i-1}) : z_i]_n = (z_1, \dots, z_{i-1})_n$ for $n \geq s + i$, $i = 1, \dots, v$. The first main result of this paper is the following characterization of $a^*(S)$.

THEOREM 2.2. *Let Q be an arbitrary reduction of S_+ generated by forms of degree* 1. *Let $\mathbf{z}$ be any filter-regular sequence of homogeneous elements of degree* 1 *which generates Q. Then*

$$a^*(S) = \max\{s(\mathbf{z}), r_Q(S_+) - v\}.$$

This result has some interesting consequences. First, we can show that

$$a^*(S) \geq -\operatorname{grade} S_+.$$

Then we can recover the formula

$$a^*(S) = \max\{a^i_{S_+}(S) \mid i \geq 0\}$$

which was recently proved by Hyry [Hy] by spectral sequence and local duality. A very closely related result was proved earlier by Johnston and Katz [JK]. Moreover, we obtain the following relationship between $a^*(S)$ and $\operatorname{reg}(S)$,

$$a^*(S) + \operatorname{grade} S_+ \leq \operatorname{reg}(S) \leq a^*(S) + l(S_+),$$

where $l(S_+)$ is the analytic spread of S_+.

The second main result of this paper is the following characterization of $a^*(G)$.

THEOREM 3.6. *Let J be an arbitrary reduction of I. Let $x_1, \dots, x_v$ be a generating sequence of J such that their images in I/I^2 form a filter-regular regular sequence of G. Then $a^*(G)$ is the least number $s \geq r_J(I) - v$ such that*

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^{s+i} = (x_1, \dots, x_{i-1}) I^{s+i-1}, \qquad i = 1, \dots, v.$$

Without loss of generality we may assume that the residue field of A is infinite. Then such a generating sequence for J always exists. Thus, Theorem 3.6 gives an effective tool for computing $a^*(G)$ because we only need to check a finite number of relations between elements of I. As applications we will compute $a^*(G)$ for some classes of ideals having small analytic deviation which occur in the works of Marley [M], Huckaba and Huneke [HH], and Aberbach and Huckaba [AH]. Especially, we can add the following new result to Huneke's beautiful theory on ideals I generated by d-sequences [Hu1, Hu3]: $a^*(G) = -\operatorname{grade} I$. This class of ideals includes, for instance, ideals generated by the maximal minors of a generic $n \times (n+1)$ matrix or by the $2n$-pfaffians of a generic $(2n+1) \times (2n+1)$ alternating matrix.

2. THE a^*-INVARIANT OF A GRADED RING

We keep the assumption and the notations of the preceding section. Moreover, for any graded S-module E, we define $\delta(E)$ to be the largest non-vanishing degree of E, where $\delta(E) = -\infty$ if $E = 0$.

If $S_n = 0$ for n large, the a^*-invariant $a^*(S)$ has been computed in [T2, Lemma 2.1]. For the sake of completeness we repeat this result here.

LEMMA 2.1. *Assume that $S_n = 0$ for n large. Then $a^*(S)$ is the maximum integer r for which $S_r \neq 0$.*

Proof. If $r = 0$, S concentrates in degree 0 and the statement is obvious. If $r > 0$, using induction on r we may assume that $H_M^j(S/S_r)_n = 0$ for $n \geq r$, $j \geq 0$. Since $H_M^j(S_r)_n = 0$ for $n \neq r$, $j \geq 0$, from the exact sequence

$$0 \to S_r \to S \to S/S_r \to 0,$$

we can deduce that $H_M^j(S)_n = 0$ for $n \geq r+1$, $j \geq 0$. Similarly, since $H_M^{\dim S_r}(S_r)_r \neq 0$, we get $H_M^{\dim S_r}(S)_r \neq 0$. Hence $a^*(S) = r$.

THEOREM 2.2. *Let Q be a reduction of S_+ which is generated by forms of degree 1. Let $\mathbf{z} = z_1, \ldots, z_v$ be a filter-regular sequence of homogeneous elements of degree 1 which generates Q. Then*

$$a^*(S) = \max\{s(\mathbf{z}), r_Q(S_+) - v\}.$$

Proof. If $v = 0$, $Q = 0$, and $S_n = 0$ for n large. In this case, $s(\mathbf{z}) = -\infty$ and $r_Q(S_+)$ is the maximum number r for which $S_r \neq 0$. Hence the conclusion follows from Lemma 2.1.

If $v > 0$, we let $S' = S/z_1S$ and $Q' = Q/z_1S$. Then Q' is a reduction of S'_+ which is generated by the sequence $\mathbf{z}' = z'_2, \ldots, z'_v$ of the images of $z_2, \ldots, z_v$ in S'. It is easily seen that $\mathbf{z}'$ is a filter-regular sequence of S' of homogeneous elements of degree 1. Using induction on v we may assume that $a^*(S') = \max\{s(\mathbf{z}'), r_Q(S'_+) - v + 1\}$. Since $s(\mathbf{z}) = \max\{\delta(0\colon z_1), s(\mathbf{z}') - 1\}$ and $r_Q(S_+) = r_{Q'}(S'_+)$, we obtain

$$\begin{aligned}\max\{s(\mathbf{z}), r_Q(S_+) - v\} &= \max\{\delta(0\colon z_1), s(\mathbf{z}') - 1, r_{Q'}(S'_+) - v\}\\ &= \max\{\delta(0\colon z_1), a^*(S') - 1\}.\end{aligned}$$

It remains to show that $a^*(S) = \max\{\delta(0\colon z_1), a^*(S') - 1\}$.

Let $n > \delta(0 : z_1)$. Since $H_M^i(0 : z_1)_n = 0$ for all i, the exact sequence

$$0 \to 0 : z_1 \to S \to S/0 : z_1 \to 0$$

implies $H_M^i(S)_n \cong H_M^i(S/0:z_1)_n$. Now, from the exact sequence

$$0 \to S/0:z_1 \xrightarrow{z_1} S \to S' \to 0$$

we can derive the exact sequence

$$\cdots \to H_M^{i-1}(S')_{n+1} \to H_M^i(S)_n \to H_M^i(S)_{n+1} \to \cdots.$$

If $n \geq a^*(S')$, $H_M^{i-1}(S')_{n+1} = 0$. Since $H_M^i(S)_n = 0$ for n large, we can deduce that $H_M^i(S)_n = 0$ for $n \geq a^*(S')$. Thus, $a^*(S) \leq \max\{\delta(0:z_1), a^*(S') - 1\}$.

To prove the converse inequality we put $a = \delta(0\colon z_1)$ and $b = a^*(S')$. If $a < b$, $H_M^i(S)_b = 0$ for all i. Hence $H_M^{i-1}(S')_b \cong H_M^i(S)_{b-1}$. Since there is an index i such that $H_M^{i-1}(S')_b \neq 0$, $H_M^i(S)_{b-1} \neq 0$. So we obtain $a^*(S) \geq b - 1 = \max\{a, b-1\}$. If $a \geq b$, we will prove that $a^*(S) \geq a$. Assume for the contrary that $a^*(S) < a$. Then $H_M^i(S)_a = 0$ and $H_M^i(S)_{a+1} = 0$ for all i. Thus, from the above exact sequences we can see that

$$H_M^{i-1}(S')_{a+1} \cong H_M^i(S/0:z_1)_a \cong H_M^{i+1}(0:z_1)_a = H_{\mathfrak{m}}^{i+1}((0:z_1)_a),$$

where $\mathfrak{m}$ denotes the maximal ideal of S_0. Put $i = \dim(0\colon z_1)_a - 1$. Then $H_{\mathfrak{m}}^{i+1}((0:z_1)_a) \neq 0$, whence $H_M^{i-1}(S')_{a+1} \neq 0$. From this it follows that $b \geq a + 1$, which contradicts the assumption. So we obtain $a^*(S) \geq a = \max\{a, b-1\}$. ∎

Remark. It was already shown in [T2] that $a^*(S) \leq \max\{s(\mathbf{z}), r_Q(S_+) - v\}$. This inequality was used in [AHT] to study the Cohen–Macaulayness of the Rees algebra of an ideal, where $s(\mathbf{z})$ is called the *sliding regularity* of $\mathbf{z}$.

COROLLARY 2.3. $a^*(S) \geq -\operatorname{grade} S_+$.

Proof. Let $g = \operatorname{grade} S_+$. Choose a generating sequence $\mathbf{z} = z_1, \ldots, z_v$ of homogeneous elements of degree 1 for S_+ (not necessarily minimal) such that $\mathbf{z}$ is filter-regular and $v > g$. Then $z_1, \ldots, z_g$ is a regular sequence and $(z_1, \ldots, z_g)\colon z_{g+1} \neq (z_1, \ldots, z_g)$. This implies $s(\mathbf{z}) \geq -g$. Hence the conclusion follows from Theorem 2.2. ∎

Now we will discuss the relationship between $a^*(S)$ and $\operatorname{reg}(S)$. For short we set

$$a_i(S) := a_{S_+}^i(S).$$

We shall need the following observations.

LEMMA 2.4. *Let z be a homogeneous filter-regular element of degree* 1 *of S. Then*

$$a_0(S) = \delta(0:z) = \delta(0:S_+).$$

Proof. Let $a = a_0(S)$. Since $H^0_{S_+}(S) = \bigcup_{n\geq 1} 0:(S_+)^n$, we have

$$H^0_{S_+}(S)_a \subseteq 0:S_+ \subseteq H^0_{S_+}(S).$$

From this it follows that $\delta(0:S_+) = a$. By the definition of a filter-regular sequence, $0:z$ is contained in the intersection of all primary components of the zero-ideal whose associated primes do not contain S_+. Since this intersection is exactly $\bigcup_{n\geq 1} 0:(S_+)^n$,

$$0:S_+ \subseteq 0:z \subseteq H^0_{S_+}(S).$$

So we obtain $a = \delta(0:S_+) \leq \delta(0:z) \leq a$. Hence $\delta(0:z) = a$. ∎

More generally, the invariant $\max\{a_i(S) \mid i = 0,\dots,v\}$, where $v \geq 0$ is any fixed integer, can be computed by means of a sequence of v homogeneous elements of degree 1.

THEOREM 2.5. *Let $\mathbf{z} = z_1,\dots,z_v$ be an arbitrary filter-regular sequence of homogeneous elements of degree* 1. *Then*

$$\begin{aligned}\max\{a_i(S) \mid i = 0,\dots,v\}\\ &= \max\{\delta((z_1,\dots,z_i):S_+/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v\}.\end{aligned}$$

Proof. By Lemma 2.4 it suffices to show that

$$\max\{a_i(S) \mid i = 0,\dots,v\} = \max\{a_0(S/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v\}.$$

By [T1, Lemma 2.1] we have

$$\begin{aligned}a_{j+1}(S/(z_1,\dots,z_{i-1})) + 1\\ &\leq a_j(S/(z_1,\dots,z_i))\\ &\leq \max\{a_j(S/(z_1,\dots,z_{i-1})), a_{j+1}(S/(z_1,\dots,z_{i-1})) + 1\},\end{aligned}$$

$i = 1,\dots,v$. Using these inequalities we can deduce that

$$a_i(S) + i \leq a_0(S/(z_1,\dots,z_i)) \leq \max\{a_j(S) + j \mid 0 \leq j \leq i\}.$$

From this it follows that

$$\begin{aligned}\max\{a_i(S) \mid i = 0,\dots,v\} &\le \max\{a_0(S/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v\}\\ &\le \max\{a_j(S) + j - i \mid 0 \le j \le i, i = 0,\dots,v\}\\ &= \max\{a_i(S) \mid i = 0,\dots,v\}.\end{aligned}$$

The conclusion is now obvious. ∎

Using Lemma 2.4 we may rewrite

$$\begin{aligned}s(\mathbf{z}) &= \max\{\delta((z_1,\dots,z_i) : z_{i+1}/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v-1\}\\ &= \max\{\delta((z_1,\dots,z_i) : S_+/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v-1\}.\end{aligned}$$

Therefore, from Theorem 2.5 we immediately obtain the following result.

COROLLARY 2.6. *Let $\mathbf{z}$ be a filter-regular sequence of v homogeneous elements of degree* 1 *of S. Then*

$$\max\{a_i(S) \mid i = 0,\dots,v-1\} = s(\mathbf{z}).$$

Now we consider the case $\mathbf{z}$ generates a reduction Q of S_+. Let $l(S_+)$ denote the *analytic spread* of S_+ which may be defined as the minimum number of generators of minimal reductions of S_+ [NR]. Then $l(S_+) \le v$. It is well known that $H^i_{S_+}(S) = 0$ for $i > l(S_+)$. Therefore, we may write

$$\max\{a_i(S) \mid i \ge 0\} = \max\{a_i(S) \mid i \le v\}.$$

COROLLARY 2.7. *Let Q be a reduction of S_+ which is generated by forms of degree* 1. *Let $\mathbf{z} = z_1,\dots,z_v$ be a filter-regular sequence of homogeneous elements of degree* 1 *which generates Q. Then*

$$\max\{a_i(S) \mid i \ge 0\} = \max\{s(\mathbf{z}), r_Q(S_+) - v\}.$$

Proof. By Theorem 2.5 and Corollary 2.6 we have

$$\begin{aligned}&\max\{a_i(S) \mid i \le v\}\\ &\quad= \max\{\delta((z_1,\dots,z_i) : S_+/(z_1,\dots,z_i)) - i \mid i = 0,\dots,v\}\\ &\quad= \max\{s(\mathbf{z}), \delta(Q : S_+/Q) - v\}.\end{aligned}$$

By Lemma 2.4, $\delta(Q : S_+/Q) = a_0(S/Q)$. Since $Q_n = S_n$ for n large, $(S_+)^n \subset Q$ for n large. Therefore, $H^0_{S_+}(S/Q) = S/Q$. Hence $a_0(S/Q)$ is the largest integer n such that $(S/Q)_n \ne 0$ which is exactly $r_Q(S_+)$. ∎

Remark. In the case $\mathbf{z}$ generates a minimal reduction Q of S_+, Corollary 2.6 has been proved by Korb [K, Lemma 4.33] and Corollary 2.7 by Herrmann *et al.* [HHK, Lemma 3.5].

Combining Theorem 2.2 with Corollary 2.7 we immediately obtain the following result of Hyry [Hy, Lemma 2.3] which was proved by spectral sequence and local duality. A very closely related result was obtained earlier by Johnston and Katz [JK, Proposition 2.1].

COROLLARY 2.8. $a^*(S) = \max\{a_i(S) \mid i \geq 0\}$

Combining the characterizations of $a^*(S)$ and $\operatorname{reg}(S)$ in terms of a reduction Q of S_+ we obtain the following relationship.

COROLLARY 2.9. $a^*(S) + \operatorname{grade}(S_+) \leq \operatorname{reg}(S) \leq a^*(S) + l(S_+)$.

Proof. Put $g = \operatorname{grade}(S_+)$ and $l = l(S_+)$. Let Q be a minimal reduction of S_+ which is generated by a filter-regular sequence $\mathbf{z} = z_1, \dots, z_l$ of homogeneous elements of degree 1. Then $g \leq l$ and $z_1, \dots, z_g$ is a regular sequence. Therefore,

$$(z_1, \dots, z_{i-1}) : z_i = (z_1, \dots, z_{i-1}), \qquad i = 1, \dots, g.$$

From this it follows that $s(\mathbf{z}) + g \leq a(\mathbf{z})$. We have

$$\begin{aligned} a^*(S) + g &= \max\{s(\mathbf{z}), r_Q(S_+) - v\} + g \leq \max\{a(\mathbf{z}), r_Q(S_+) - l + g\} \\ &\leq \max\{a(\mathbf{z}), r_Q(S)\} = \operatorname{reg}(S). \end{aligned}$$

Since $\operatorname{reg}(S) = \max\{a_i(S) + i \mid i \geq 0\}$ and $a^*(S) = \max\{a_i(S) \mid i \geq 0\}$, we get

$$\operatorname{reg}(S) \leq a^*(S) + l.$$

∎

3. THE a^*-INVARIANT OF THE ASSOCIATED GRADED RING

Let $(A, \mathfrak{m})$ be a local ring and I an ideal of A with $\operatorname{ht} I > 0$. Recall that an ideal J is a *reduction* of I if there is an integer n such that $JI^n = I^{n+1}$. The least integer n with this property is the *reduction number* of I with respect to J, denoted by $r_J(I)$. Let $G = \oplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of I. Given an element x of I we will denote by x^* the image of x in $I/I^2 \subset G$. Moreover, we set $I^n = A$ if $n \leq 0$.

In [T3] we have given the following characterization for reg(G) in terms of any reduction J of I: Let $x_1, \dots, x_v$ be a generating sequence of J such that $\mathbf{z} = x_1^*, \dots, x_v^*$ is a filter-regular sequence of G. Then reg(G) is the least integer $s \geq r_J(I)$ such that

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^{s+1} = (x_1, \dots, x_{i-1}) I^s, \qquad i = 1, \dots, v.$$

This gives an effective tool for estimating reg(G) because we only need to check a finite number of relations between $x_1, \dots, x_v$. Note that if the residue field of A is infinite, any reduction J of I has a basis $x_1, \dots, x_v$ such that $\mathbf{z}$ is a filter-regular sequence of G (see [T1, Lemma 3.1]).

The aim of this section is to find a similar characterization for $a^*(G)$ in terms of any reduction of I. We begin with the following observation.

LEMMA 3.1. *Let $J = (x_1, \dots, x_v)$ be an arbitrary reduction of I and $Q = (x_1^*, \dots, x_v^*)$. Then Q is a reduction of G_+ and $r_Q(G_+) = r_J(I)$.*

Proof. It is easily seen that $Q = \oplus_{n \geq 1}(JI^{n-1} + I^{n+1}/I^{n+1})$. For $n \geq r_J(I)$, we have $JI^n = I^{n+1}$ and therefore $Q_{n+1} = I^{n+1}/I^{n+2} = G_{n+1}$. Thus, Q is a reduction of G_+ and $r_Q(G_+) \leq r_J(I)$. Conversely, if $Q_{n+1} = G_{n+1}$, then $JI^n + I^{n+2} = I^{n+1}$. By Nakayama's Lemma, this implies $JI^n = I^{n+1}$. Hence $r_Q(G_+) \geq r_J(I)$. ∎

By Lemma 3.1 and Theorem 2.2 we have

$$a^*(G) = \max\{s(\mathbf{z}), r_Q(G_+) - v\} = \max\{s(\mathbf{z}), r_J(I) - v\}.$$

It remains to estimate $s(\mathbf{z})$ in terms of $x_1, \dots, x_v$. For this we need to study properties of a filter-regular sequence of forms of degree 1 in G.

LEMMA 3.2. *Let x be an arbitrary element in I such that x^* is a filter-regular element of G. Then $(x) \cap I^n = xI^{n-1}$ for $n >$ $\max\{a_0(G), a_1(G) + 1\}$.*

Proof. Consider the ideal $K := \oplus_{n \geq 0}((x) \cap I^n + I^{n+1})/(xI^{n-1} + I^{n+1})$ of the quotient ring $G/(x^*) = \oplus_{n \geq 0} I^n/I^{n+1} + xI^{n-1}$. By the Artin–Rees lemma, there is an integer $c \geq 2$ such that $(x) \cap I^n \subseteq xI^{n-c}$ and therefore

$$(x) \cap I^n = (x) \cap I^n \cap xI^{n-c} = x[(I^n : x) \cap I^{n-c}]$$

for $n \geq c$. Since x^* is filter-regular, $(0 : x^*)_m = 0$ and therefore $(I^{m+2} : x) \cap I^m = I^{m+1}$ for m large. From this it follows that for $n \geq m + 2$,

$$(I^n : x) \cap I^m = (I^n : x) \cap (I^{m+2} : x) \cap I^m = (I^n : x) \cap I^{m+1}.$$

Thus, for n large we have

$$(I^n : x) \cap I^{n-c} = (I^n : x) \cap I^{n-c+1} = \cdots = (I^n : x) \cap I^{n-1} = I^{n-1}.$$

So we obtain $(x) \cap I^n = xI^{n-1}$ and therefore $K_n = 0$ for n large enough. As a consequence, $H^0_{G_+}(K) = K$. Hence K is a submodule of $H^0_{G_+}(G/(x^*))$. Let $n > a_0(G/(x^*))$. Then $K_n = 0$. That is, $I^{n+1} + (x) \cap I^n = I^{n+1} + xI^{n-1}$. Hence $(x) \cap I^n = xI^{n-1} + (x) \cap I^{n+1}$. From this it follows that

$$(x) \cap I^n = xI^{n-1} + \bigcap_{m \geq n} (x) \cap I^{m+1} = xI^{n-1}.$$

By [T1, Lemma 2.3] we know that $a_0(G/(x^*)) \leq \max\{a_0(G), a(G) + 1\}$. ∎

Remark. Lemma 3.2 is a generalization of [T3, Lemma 4.4(i)] where we proved the same relation for $n > \operatorname{reg}(G)$.

PROPOSITION 3.3. *Let $x_1, \ldots, x_v$ be an arbitrary sequence of elements of I such that $\mathbf{z} = x_1^*, \ldots, x_v^*$ is a filter-regular sequence in G. Then*

$$[(x_1, \ldots, x_{i-1}) : x_i] \cap I^n = (x_1, \ldots, x_{i-1}) I^{n-1}$$

for $n \geq s(\mathbf{z}) + i$, $i = 1, \ldots, v$,

Proof. By definition, $s(\mathbf{z})$ is the least integer s such that

$$[(x_1^*, \ldots, x_{i-1}^*) : x_i^*]_n = (x_1^*, \ldots, x_{i-1}^*)_n$$

for $n \geq s + i$, $i = 1, \ldots, v$. Translating these relations in terms of $x_1, \ldots, x_v$ we get

$$\{[(x_1, \ldots, x_{i-1}) I^n + I^{n+2}] : x_i\} \cap I^n = (x_1, \ldots, x_{i-1}) I^{n-1} + I^{n+1} \quad (*)$$

for $n \geq s(\mathbf{z}) + i$, $i = 1, \ldots, v$. We shall see that the conclusion follows from these relations.

For $i = 1$ let $n \geq s(\mathbf{z}) + 1$. Then $(I^{n+2} : x_1) \cap I^n = I^{n-1}$ by $(*)$. This implies

$$(0 : x_1) \cap I^n = (0 : x_1) \cap \left(I^{n+2} : x_1\right) \cap I^n = (0 : x_1) \cap I^{n+1}.$$

Therefore, $(0 : x_1) \cap I^n = \bigcap_{m \geq n} (0 : x_1) \cap I^m = 0$.

For $i > 1$ let $n \geq s(\mathbf{z}) + i$. We will first show that

$$\begin{aligned}\{[(x_2, \ldots, x_{i-1}) I^n + \left(I^{n+2}, x_1\right)] : x_i\} &\cap (I^n, x_1) \\ &= (x_2, \ldots, x_{i-1}) I^{n-1} + \left(I^{n+1}, x_1\right),\end{aligned}$$

which is just the relation $(*)$ formulated for the residue classes of $x_2, \dots, x_v$ in $A/(x_1)$. It is obvious that the right side is contained in the left side. To prove the converse inclusion let $a + bx_1$ be an arbitrary element of $\{[(x_2, \dots, x_{i-1})I^n + (I^{n+2}, x_1)] : x_i\}$. That is, $a \in I^n$ and $(a + bx_1)x_i \in (x_2, \dots, x_{i-1})I^n + (I^{n+2}, x_1)$. Then there is an element $c \in A$ such that $(a + bx_1)x_i - cx_1 \in (x_2, \dots, x_{i-1})I^n + I^{n+2}$. It follows that $(bx_i - c)x_1 \in (x_1) \cap I^{n+1}$. By Corollary 2.6, $n \geq s(\mathbf{z}) + i > \max\{a_0(G), a_1(G)\} + 1$. Hence $(x_1) \cap I^{n+1} = x_1 I^n$ by Lemma 3.2. Thus, $(bx_i - c)x_1$ and therefore ax_i belongs to $(x_1, \dots, x_{i-1})I^n + I^{n+2}$. Using $(*)$ we get $a \in (x_1, \dots, x_{i-1})I^{n-1} + I^{n+1}$. So $a + bx_1 \in (x_2, \dots, x_{i-1})I^{n-1} + (I^{n+1}, x_1)$.

Now, we may apply the induction hypothesis to $A/(x_1)$ and obtain

$$[(x_1, \dots, x_{i-1}) : x_i] \cap (I^n, x_1) = (x_1) + (x_2, \dots, x_{i-1})I^{n-1}.$$

Since $(x_1) \cap I^n = x_1 I^{n-1}$ by Lemma 3.2, we can conclude that

$$\begin{aligned}[(x_1, \dots, x_{i-1}) : x_i] \cap I^n &= (x_1) \cap I^n + (x_2, \dots, x_{i-1})I^{n-1} \\ &= (x_1, \dots, x_{i-1})I^{n-1}.\end{aligned}$$

∎

PROPOSITION 3.4. *Let $J = (x_1, \dots, x_v)$ be a reduction of I and $\mathbf{z} = x_1^*, \dots, x_v^*$. Let $s \geq r_J(I) - v$ be an integer such that*

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^{s+i} = (x_1, \dots, x_{i-1})I^{s+i-1}$$

for $i = 1, \dots, v$. Then

(i) $[(x_1, \dots, x_{i-1}) : x_i] \cap I^n = (x_1, \dots, x_{i-1})I^{n-1}$ *for* $n \geq s + i$, $i = 1, \dots, v$.

(ii) $s(\mathbf{z}) \leq s$.

Proof. (i) By the assumption we only need to check the case $n > s + i$, $i = 1, \dots, v$.

For $i = v$ let $n > s + v$. Then $n > r_J(I)$. Hence $I^n = (x_1, \dots, x_v)I^{n-1}$. From this is follows that

$$\begin{aligned}[(x_1, \dots, x_{v-1}) : x_v] \cap I^n = {} & (x_1, \dots, x_{v-1})I^{n-1} \\ & + x_v\{[(x_1, \dots, x_{v-1}) : x_v^2] \cap I^{n-1}\}.\end{aligned}$$

By induction on n we may assume that $[(x_1,\dots,x_{v-1}):x_v]\cap I^{n-1}=(x_1,\dots,x_{v-1})I^{n-2}$. Then $[(x_1,\dots,x_{v-1}):x_v^2]\cap(I^{n-1}:x_v)\subseteq(x_1,\dots,x_{v-1}):x_v$. Hence

$$\begin{aligned}\left[(x_1,\dots,x_{v-1}):x_v^2\right]\cap I^{n-1}&=\left[(x_1,\dots,x_{v-1}):x_v^2\right]\cap\left(I^{n-1}:x_v\right)\cap I^{n-1}\\&\subseteq\left[(x_1,\dots,x_{v-1}):x_v\right]\cap I^{n-1}\\&=(x_1,\dots,x_{v-1})I^{n-2}.\end{aligned}$$

It follows that $x_v\{[(x_1,\dots,x_{v-1}):x_v^2]\cap I^{n-1}\}\subseteq(x_1,\dots,x_{v-1})I^{n-1}$. Thus,

$$\left[(x_1,\dots,x_{v-1}):x_v\right]\cap I^n=(x_1,\dots,x_{v-1})I^{n-1}.$$

For $i<v$ let $n>s+i$. Using the induction hypothesis on n and on i we get

$$\begin{aligned}\left[(x_1,\dots,x_{i-1}):x_i\right]\cap I^n&=\left\{\left[(x_1,\dots,x_{i-1}):x_i\right]\cap I^{n-1}\right\}\cap I^n\\&=(x_1,\dots,x_{i-1})I^{n-2}\cap I^n\\&\subseteq\left[(x_1,\dots,x_i):x_{i+1}\right]\cap I^n=(x_1,\dots,x_i)I^{n-1}.\end{aligned}$$

From this it follows that

$$\begin{aligned}\left[(x_1,\dots,x_{i-1}):x_i\right]\cap I^n=(x_1,\dots,x_{i-1})I^{n-1}&\\+x_i\left\{\left[(x_1,\dots,x_{i-1}):x_i^2\right]\cap I^{n-1}\right\}&.\end{aligned}$$

Now we can proceed as for $i=v$ and obtain

$$\left[(x_1,\dots,x_{i-1}):x_i\right]\cap I^n=(x_1,\dots,x_{i-1})I^{n-1}.$$

(ii) We have to prove that $[(x_1^*,\dots,x_{i-1}^*):x_i^*]_n=(x_1^*,\dots,x_{i-1}^*)_n$ or, equivalently,

$$\left\{\left[(x_1,\dots,x_{i-1})I^n+I^{n+2}\right]:x_i\right\}\cap I^n=(x_1,\dots,x_{i-1})I^{n-1}+I^{n+1}$$

for $n\ge s+i$, $i=1,\dots,v$.

For $i=1$ let $n\ge s+1$. If $v=1$, $s\ge r_J(I)-1$. Then $I^{n+2}=x_1I^{n+1}$. Hence $I^{n+2}:x_1=I^{n+1}+(0:x_1)$. If $v>1$, using (i) we get $[(x_1):x_2]\cap I^m=x_1I^{m-1}$ and therefore

$$(x_1)\cap I^m=x_1I^{m-1}$$

for $m\ge s+2$. As a consequence, $I^{n+2}:x_1=I^{n+1}+(0:x_1)$. Thus, using (i) we obtain

$$\left(I^{n+2}:x_1\right)\cap I^n=I^{n+1}+(0:x_1)\cap I^n=I^{n+1}.$$

For $i > 1$ we consider the quotient ring $\overline{A} = A/(x_1)$ and the ideal $\bar{I} = I/(x_1)$. Let $\bar{x}_2, \ldots, \bar{x}_v$ denote the images of $x_2, \ldots, x_v$ in $\overline{A}$. Then the ideal $\bar{J} = (\bar{x}_2, \ldots, \bar{x}_v)$ is a reduction of $\bar{I}$ with $r_{\bar{J}}(\bar{I}) \leq r_J(I)$. From the assumption we get

$$[(x_1, \ldots, x_{i-1}) : x_i] \cap \left(I^{s+i}, x_1\right) = (x_1) + (x_2, \ldots, x_{i-1})I^{s+i-1}$$

for $i = 2, \ldots, v$. This means $[(\bar{x}_2, \ldots, \bar{x}_{i-1}) : \bar{x}_i] \cap \bar{I}^{s+i} = (\bar{x}_2, \ldots, \bar{x}_{i-1})\bar{I}^{s+i-1}$. Since $s + 1 \geq r_J(I) - v + 1 = r_{\bar{J}}(\bar{I}) - (v-1)$, using induction on v we may assume that

$$\left\{\left[(\bar{x}_2, \ldots, \bar{x}_{i-1})\bar{I}^n + \tilde{I}^{n+2}\right] : \bar{x}_i\right\} \cap \bar{I}^n = (\bar{x}_2, \ldots, \bar{x}_{i-1})\bar{I}^{n-1} + \bar{I}^{n+1}$$

for $n \geq s + i$. Translating this relation in terms of I we get

$$\begin{aligned}\left\{\left[(x_1) + (x_2, \ldots, x_{i-1})I^n + I^{n+2}\right] : x_i\right\} &\cap \left(I^n, x_1\right) \\ &= (x_1) + (x_2, \ldots, x_{i-1})I^{n-1} + I^{n+1},\end{aligned}$$

for $n \geq s + i$, which implies

$$\begin{aligned}\left\{\left[(x_1, \ldots, x_{i-1})I^n + I^{n+2}\right] : x_i\right\} &\cap I^n \\ &= (x_1) \cap I^n + (x_2, \ldots, x_{i-1})I^{n-1} + I^{n+1} \\ &= (x_1, \ldots, x_{i-1})I^{n-1} + I^{n+1}.\end{aligned}$$

∎

COROLLARY 3.5. *Let $J = (x_1, \ldots, x_v)$ be a reduction of I. Then $\mathbf{z} = x_1^*, \ldots, x_v^*$ is a filter-regular sequence if and only if there is an integer $s \geq r_J(I) - v$ such that*

$$[(x_1, \ldots, x_{i-1}) : x_i] \cap I^{s+i} = (x_1, \ldots, x_{i-1})I^{s+i-1}, \qquad i = 1, \ldots, v.$$

Proof. If $\mathbf{z}$ is filter-regular, the above relations hold for $s = \max\{s(\mathbf{z}), r_J(I) - v\}$ by Proposition 3.3. Conversely, assume that the above relations hold for an integer $s \geq r_J(I) - v$. Then $s(\mathbf{z}) \leq s$ by Proposition 3.4. Hence $a(\mathbf{z}) < \infty$. By [T1, Lemma 2.1], this implies that $\mathbf{z}$ is a filter-regular sequence. ∎

Now we are able to characterize $a^*(G)$ in terms of any reduction of I.

THEOREM 3.6. *Let $J = (x_1, \ldots, x_v)$ be a reduction of I. Assume that $\mathbf{z} = x_1^*, \ldots, x_v^*$ is a filter-regular sequence. Then $a^*(G)$ is the least integer $s \geq r_J(I) - v$ such that*

$$[(x_1, \ldots, x_{i-1}) : x_i] \cap I^{s+i} = (x_1, \ldots, x_{i-1})I^{s+i-1}, \qquad i = 1, \ldots, v.$$

Proof. Let Q denote the ideal in G generated by $\mathbf{z}$. By Lemma 3.1, Q is a reduction of G_+ with $r_Q(S_+) = r_J(I)$. Applying Theorem 2.2 we obtain

$$a^*(G) = \max\{s(\mathbf{z}), r_J(I) - v\}.$$

By Proposition 3.3, $\max\{s(\mathbf{z}), r_J(I) - v\} \le s$. On the other hand, Proposition 3.4 gives $\max\{s(\mathbf{z}), r_J(I) - v\} \le s$. Thus,

$$\max\{s(\mathbf{z}), r_J(I) - v\} = s.$$

∎

For later applications we give the following consequence of Theorem 3.6.

COROLLARY 3.7. *Let* $g = \operatorname{grade} I$ *and* $J = (x_1, \dots, x_v)$ *be a reduction of* I. *Assume that*

(i) $x_1, \dots, x_g$ *is a regular sequence,*
(ii) $[(x_1, \dots, x_{i-1}) : x_i] \cap I^{i-g} = (x_1, \dots, x_{i-1})I^{i-g-1}$, $i = g + 1, \dots, v$,
(iii) $r_J(I) \le v - g$.

Then $a^*(G) = -g$ *and* $\operatorname{grade} G_+ = g$.

Proof. The conditions (i) and (ii) imply that

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^{i-g} = (x_1, \dots, x_{i-1})I^{i-g-1}$$

for $i = 1, \dots, v$. Condition (iii) gives $-g \ge r_J(I) - v$. If $v = g$, $r_J(I) = 0$. If $v > g$,

$$[(x_1, \dots, x_g) : x_{g+1}] \cap I^n = (x_1, \dots, x_g)I^{n-1}$$

for $n \ge 1$ by Proposition 3.4. Since $x_1, \dots, x_{g+1}$ is not a regular sequence, $(x_1, \dots, x_g) : x_{g+1} \ne (x_1, \dots, x_g)$. Thus, $-g$ is the last integer $s \ge r_J(I) - v$ such that

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^{s+i} = (x_1, \dots, x_{i-1})I^{s+i-1}$$

for $i = 1, \dots, v$. By Corollary 3.5 and Theorem 3.6 we get $a^*(G) = -g$. By Corollary 2.3, this implies $\operatorname{grade} G_+ \ge g$. Since $\operatorname{grade} G_+ \le \operatorname{grade} I = g$, we obtain $\operatorname{grade} G_+ = g$. ∎

4. APPLICATIONS

In this section we shall apply Theorem 3.6 to compute the a^*-invariant of the associated graded rings of some classes of ideals.

Let $(A, \mathfrak{m})$ be a local ring and I an ideal of I. For convenience we assume that the residue field of A is infinite. As before, we will denote by G the associated graded ring of I. Let $l(I) = \dim G/\mathfrak{m}G$ be the analytic spread of I. Then $l(I)$ is the minimum number of generators of a minimal reduction of I [NR]. The difference $l(I) - \operatorname{ht} I$ is called the *analytic deviation* of I, denoted by $\operatorname{ad}(I)$ [HH]. If $l(I) = \operatorname{ht} I$, I is called an *equimultiple* ideal.

PROPOSITION 4.1. *Let I be an equimultiple ideal with* grade $I = \operatorname{ht} I = g$. *Assume that* grade $G_+ \geq g - 1$. *Let J be an arbitrary minimal reduction of I. Then*

$$a^*(G) = a_g(G) = r_J(I) - g.$$

Proof. There is a regular sequence $x_1, \dots, x_g$ of A which minimally generates J such that the sequence of the initial forms $x_1^*, \dots, x_{g-1}^*$ is regular. By the Valabrega–Valla theorem [VV, Theorem 2.3] we have

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^n = (x_1, \dots, x_{i-1}) \cap I^n = (x_1, \dots, x_{i-1}) I^{n-1}$$

for all n, $i = 1, \dots, g$. Therefore, $a^*(G) = r_J(I) - g$ by Theorem 3.6.

On the other hand, if grade $G_+ = g$, then $a_i(G) = -\infty$ for $i = 1, \dots, g-1$ [T3, Corollary 2.6]. If grade $G_+ = g - 1$, then $a_i(G) = -\infty$ for $i = 0, \dots, g-2$ and $a_{g-1}(G) < a_g(G)$ [Ho, Theorem 5.2] (cf. [M, Theorem 2.1]). Now, applying Corollary 2.8 we obtain

$$a^*(G) = \max\{a_i(G) \mid i \geq 0\} = a_g(G).$$

∎

Proposition 4.1 is a generalization of the following result of Marley [M, Corollary 2.2].

COROLLARY 4.2. *Let A be a Cohen–Macaulay ring and I an $\mathfrak{m}$-primary ideal. Assume that* grade $G_+ \geq d - 1$, $d = \dim A$. *Let J be an arbitrary minimal reduction of I. Then*

$$a^*(G) = a(G) = r_J(I) - d.$$

We say that I is *generically a complete intersection* if $I_\wp$ is a complete intersection for every prime ideal $\wp \supseteq I$ with $\dim A/\wp = \dim A/I$.

PROPOSITION 4.3. *Let A be a Cohen–Macaulay ring and I an ideal with* $\operatorname{ht} I \geq 1$, $\operatorname{ad}(I) = 1$. *Assume that the following conditions are satisfied*:

(i) *I is generically a complete intersection,*

(ii) $r(I) \leq 1$.

Then $a^*(G) = -\operatorname{ht} I$ *and* $\operatorname{grade} G_+ = \operatorname{ht} I$.

Proof. Set $r = \operatorname{ht} I$. Let $J = (x_1, \dots, x_{r+1})$ be a minimal reduction of I with $r_J(I) \leq 1$. We may choose $x_1, \dots, x_{r+1}$ such that $x_1, \dots, x_r$ is a regular sequence which generates I generically. By [HH, Remark 2.1(iii)],

$$[(x_1, \dots, x_r) : x_{r+1}] \cap I = (x_1, \dots, x_r).$$

Thus, the conclusion follows from Corollary 3.7. ∎

COROLLARY 4.4. *Let A be a Cohen–Macaulay ring and I a prime ideal with* $\operatorname{ht} I \geq 1$, $\operatorname{ad}(I) = 1$. *Assume that A_I is a regular local ring and $I_\wp$ is a complete intersection for every prime ideal $\wp \supseteq I$ with* $\operatorname{ht} \wp/I = 1$. *Then* $\operatorname{grade} G_+ = \operatorname{ht} I$ *and* $a^*(G) = -\operatorname{ht} I$.

Proof. By [HH, Corollary 2.6], $r_J(I) \leq 1$ for every minimal reduction J of I. Therefore, the assumption of Proposition 4.3 is satisfied. ∎

Remark. Under the assumption of Proposition 4.3 and Corollary 4.4, the associated graded ring G need not to be Cohen–Macaulay. In fact, if I is an unmixed ideal which satisfies the assumption of Proposition 4.3, Huckaba and Huneke have shown that G is Cohen–Macaulay if and only if $\operatorname{depth} A/I \geq \dim A/I - 1$ [HH, Theorem 2.9].

PROPOSITION 4.5. *Let A be a Cohen–Macaulay ring and I an ideal with* $\operatorname{ht} I \geq 1$, $\operatorname{ad}(I) = 2$. *Assume that the following conditions are satisfied*:

(i) *I is generically a complete intersection,*

(ii) $r(I_\wp) \leq 1$ *for every prime $\wp \supseteq I$ with* $\operatorname{ht} \wp/I = 1$,

(iii) $r(I) \leq 2$.

Then $a^*(G) = -\operatorname{ht} I$ *and* $\operatorname{grade} G_+ = \operatorname{ht} I$.

Proof. Set $r = \operatorname{ht} I$. Let $J = (x_1, \dots, x_{r+2})$ be a minimal reduction of I with $r_J(I) \leq 2$. By [AH, Proposition 2.2] we may choose $x_1, \dots, x_{r+2}$ such that $x_1, \dots, x_r$ is a regular sequence and

$$[(x_1, \dots, x_r) : x_{r+1}] \cap I = (x_1, \dots, x_r),$$

$$[(x_1, \dots, x_{r+1}) : x_{r+2}] \cap I^2 = (x_1, \dots, x_{r+1})I.$$

Therefore, the conclusion follows from Corollary 3.7. ∎

Recall that $x_1, \dots, x_v$ is called a *d-sequence* if

(i) $x_i \notin (x_1, \dots, x_{i-1})$.

(ii) $(x_1, \dots, x_{i-1}) : x_i x_k = (x_1, \dots, x_{i-1}) : x_k$

for $k \geq i$, $i = 1, \dots, v$. This notion was introduced by Huneke who has showed that ideals generated by d-sequences enjoy many interesting properties and that there are many interesting classes of ideals generated by d-sequences [Hu1–Hu3].

PROPOSITION 4.6. *Let I be an ideal generated by a d-sequence. Then $a^*(G) = -\operatorname{grade} I$.*

Proof. Set $g = \operatorname{grade} I$. Let $x_1, \dots, x_v$ be a d-sequence generating I. Then $x_1, \dots, x_g$ is a regular sequence. Hence

$$(x_1, \dots, x_{i-1}) : x_i = (x_1, \dots, x_{i-1}), \qquad i = 1, \dots, g.$$

From [Hu2, Proof of Theorem 2.1] we can deduce that

$$[(x_1, \dots, x_{i-1}) : x_i] \cap I^n = (x_1, \dots, x_{i-1}) I^{n-1}$$

for $n \geq 1$, $i = 1, \dots, v$. Since $r_I(I) = 0$, applying Corollary 3.7 we get $a^*(G) = -g$. ∎

It is known that the ideals of the following corollaries are generated by d-sequences [Hu3] and that their associated graded rings G are Cohen–Macaulay [Hu2, EH], whence $a^*(G) = a(G)$.

COROLLARY 4.7. *Let A be a regular local ring and I an ideal linked to a Gorenstein prime ideal. Then $a(G) = -\operatorname{ht} I$.*

COROLLARY 4.8. *Let k be a field and X a generic $n \times (n+1)$ matrix. Let I be the ideal in $k[X]$ generated by the maximal minors of X. Then $a(G) = -2$.*

COROLLARY 4.9. *Let k be a field and X a generic $(2n+1) \times (2n+1)$ alternating matrix. Let I be the ideal in $k[X]$ generated by the $2n$-Pfaffians of X. Then $a(G) = -3$.*

Remark. The above corollaries can be also deduced from the following formula for the a-invariant of a Cohen–Macaulay associated graded ring which was discovered by Ulrich [U, Theorem 1.4]:

$$a(G) = \max\{r(I_\wp) - \operatorname{ht} I_\wp \mid \wp \text{ is a prime ideal containing } I \text{ with } l(I_\wp) = \operatorname{ht} I_\wp\}.$$

See [HHK, Theorem 4.5] for a generalization of this result.

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